# IUCAA Refresher Course 2026

## Foundations of GW: Theory, Sources, and Detection

## I. Foundations of GW: Basic theory

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# LIGO observation (Nobel 2017) of GW (first direct proof)

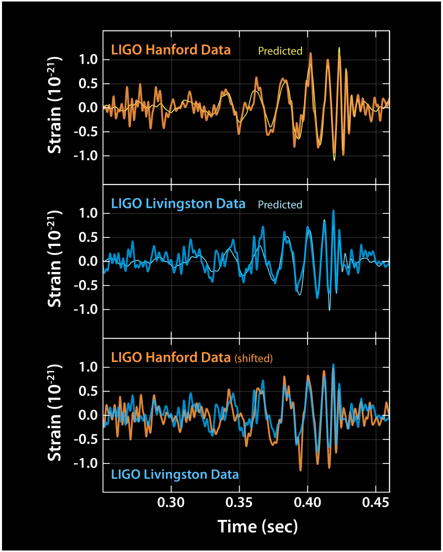


Figure 1: **GW150914: Strain signal observed at LIGO**

$$10^{-21}$$

Dimensionless, abnormally small number.

$$\frac{\Delta L}{L} \sim h(t)$$

Strain: Fractional change in effective length.

General Relativity, Interferometry

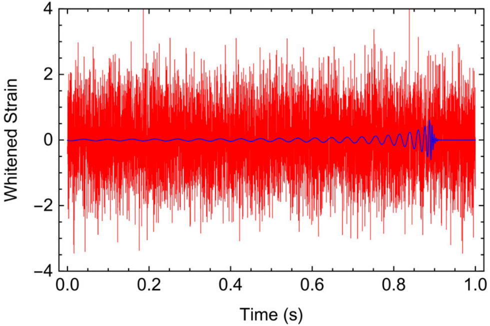


Figure 2: **GW150914: Raw signal (red) and extracted information (blue)**

## Data analysis

# Hulse-Taylor (Nobel 1993) and GWs (first, indirect proof)

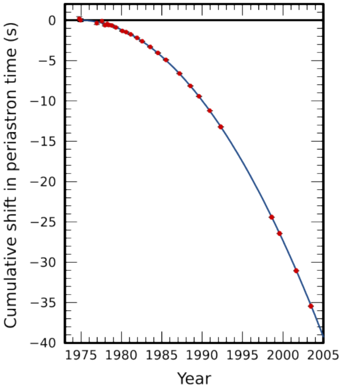


Figure 3: **Cumulative periastron shift**

$\frac{dP}{dt} = -2.4 \times 10^{-12}$ **for PSR B 1913+16(using GR and grav. radiation). Periastron (point of closest approach between two masses) shift in 20 years is around** $-17.24$**. Similar and even better results for PSR J0737-3739.**

## Timeline: pre-history

**Laplace (1825)**: speculation- speed of gravity must be at least 100 million times faster than the speed of light.

**Heaviside (1893)**: formulation of gravity analogous to electromagnetism, anticipating concepts that would later appear in the theory of relativity including the idea of gravitational waves.

**Poincare (1905)**: while developing a relativistic theory of gravity inspired by Maxwell's equations, explicitly introduced the term gravitational wave.

## Timeline: 20th century

Einstein (1916): derived a formula for gravitational radiation. It contained a major error. The radiation was monopolar, i.e., originating from an object expanding and contracting radially. Abraham and Nordström, pointed Einstein in the correct direction, but did not derive the right formula.

Einstein (1918): derived the correct quadrupole formula for gravitational wave emission but missed a factor of 2.

Eddington (1922): finally presented the correct formula. Argued that if longitudinal gravitational waves were real, they would propagate at the speed of thought, i.e., instantaneously, contradicting relativity. Concluded that only transverse ones could be physically possible. First to apply the quadrupole formula to a rotating bar and calculate energy loss via gravitational radiation.

Einstein and Rosen (1936): submitted a paper to Physical Review asserting that gravitational waves could not occur. Manuscript rejected, referee points out errors, suggests corrections. Einstein protests to editor Tate, vowing never to publish there. The referee, Robertson, discusses the errors with Infeld who convinces Einstein to redo the calculations. Corrected version published in the Journal of the Franklin Institute.

Landau and Lifshitz (1941): showed that the quadrupole formula is valid for describing the emission of gravitational waves by binary star systems in orbit.

Pirani (1957) recognizes GWs as manifestations of spacetime curvature (Riemann). Demonstrates that free-falling particles experience geodesic deviations as a gravitational wave passes (observable tidal-force). Bondi, shows that gravitational waves carry energy as they propagate (Feynman).

Bondi (1962) formally proved that systems emitting gravitational waves lose mass, following work in 1958 by Trautman.

Gertsenshtein and Pustovoit (1962) were the first to describe how Michelson interferometry could be used to detect gravitational waves.

Thorne (1967), began systematically developing the theory of gravitational waves, laying the foundation for its application to realistic astrophysical contexts.In parallel, Penrose and Newman, introduced new mathematical tools, including the Newman-Penrose formalism, which proved important for accurately describing gravitational radiation in asymptotically flat spacetimes.

Isaacson (1968) defines the gauge-invariant (approx) and conserved energy-momentum tensor for high-frequency gravitational waves, describing them as a source of energy that back-reacts on the curvature of spacetime.

Burke(1970): applied the method of matched asymptotic expansions to general relativity, enabling consistent matching between the near field of the gravitational-wave source and the far-field region where radiation is observed.

Vishveshwara (1970): discovered quasinormal modes in black hole perturbations.

Weber (1970), used resonant bar detectors and announced the detection of gravitational waves though it was not accepted.

Hawking (1971) used his area theorem (the event horizon area of black holes cannot decrease) to show that in a collision of two black holes, a maximum amount of mass can be converted into gravitational radiation. For the merger of two equal-mass, non-spinning black holes, this limit is about 29 percent of the total initial mass.

Hulse and Taylor (1974) discovered a pulsar in a compact binary system and, over the years, showed that the system's orbital radius was shrinking in agreement with energy loss via gravitational waves, as predicted by general relativity. For this indirect confirmation of the existence of gravitational waves, they received the 1993 Nobel Prize in Physics.

Weiss and Thorne (1975) discuss and develop the proposal for a laser-based Michelson interferometer for detecting gravitational waves. The U.S. NSF, approves initial funding for the project that would become LIGO. Drever, joined them, forming the troika.

Christodoulou (1991), mathematically demonstrated the existence of the gravitational wave memory effect: a permanent change in the relative separation of test particles after a wave passes.

Barish (1994), assumes leadership of the LIGO project, introducing a renewed emphasis on experimental feasibility and the practical construction of the detectors. Under his direction, construction began on two LIGO observatories: one in Livingston, Louisiana, and the other in Hanford, Washington.

## Timeline: 21st century

**2002 - LIGO** began its first observational run, named O1, aiming to detect gravitational waves. Virgo was later added to the Hanford and Livingston LIGO interferometers. Marks beginning of international collaboration.

**Pretorius (2005)** achieved the first successful numerical relativity solution of the black hole merger problem, catalyzing the development of gravitational wave templates now in use in modern interferometers.

**2015, September 14**: First detection by LIGO.
**2017 Nobel Prize** to Weiss, Barish and Thorne.

**2023**: Evidence of a low-frequency gravitational wave background detected through the analysis of signals from millisecond pulsars. Believed to originate from mergers of supermassive black holes throughout cosmic history.

## The Scalar-Vector-Tensor Decomposition of $h_{ij}$

**Metric perturbation decomposition:**

Split the metric perturbation $h_{ij}$ according to their transformation properties: $h_{tt}$ scalar under rotations, $h_{t\alpha}$ a vector and $h_{\alpha\beta}$ a tensor. Do a Helmholtz decomposition of the vector into a gradient of a scalar and a divergenceless vector. Similarly a tensor decomposed into two scalars, a divergenceless vector and a transverse, traceless tensor. Total 10 in $\phi$, $\gamma$ $\beta_\alpha$, $H$, $\epsilon_\alpha$, $\lambda$ and $h_{ij}^{TT}$.

$$h_{tt} = 2\phi \quad ; \quad h_{t\alpha} = \partial_\alpha \gamma + \beta_\alpha \ (\partial_\alpha \beta^\alpha = 0)$$

$$h_{\alpha\beta} = \frac{1}{3} H \delta_{\alpha\beta} + \partial_{(\alpha} \epsilon_{\beta)} + \left( \partial_\alpha \partial_\beta - \frac{1}{3} \delta_{\alpha\beta} \nabla^2 \right) \lambda + h_{\alpha\beta}^{TT}$$

with $\partial_\alpha \epsilon^\alpha = 0$, $\partial_\alpha h^{\alpha\beta(TT)} = 0$, $h_\alpha^{\alpha(TT)} = 0$

Decompositions are uniquely defined upto b.c. For example,

$$\partial_\alpha h_{t\alpha} = \nabla^2\gamma + \partial_\alpha\beta_\alpha = \nabla^2\gamma \to \gamma = \nabla^{-2}(\partial_\alpha h_{\alpha t})$$

**Stress tensor decomposition:**

$$T_{tt} = \rho \quad ; \quad T_{t\alpha} = \partial_\alpha S + S_\alpha \ (\partial_\alpha S^\alpha = 0)$$
$$T_{\alpha\beta} = p\delta_{\alpha\beta} + \partial_{(\alpha}\sigma_{\beta)} + \left(\partial_\alpha\partial_\beta - \frac{1}{3}\delta_{\alpha\beta}\nabla^2\right)\sigma + \sigma_{\alpha\beta}$$

with $\partial_\alpha\sigma^\alpha = 0$, $\partial_\alpha\sigma_{\alpha\beta} = 0$, $\sigma^\alpha_\alpha = 0$

Total 10 in $\rho$, $S$, $S_\alpha$, $p$, $\sigma_\alpha$, $\sigma$ and $\sigma_{\alpha\beta}$.

**Decomposition of generator $\xi^i$ of infinitesimal coordinate transformations:**

$$\xi_t = A \quad ; \quad \xi_\alpha = \partial_\alpha C + B_\alpha, \;\; (\partial_\alpha B^\alpha = 0)$$

Use in $h'_{ij} = h_{ij} - \partial_i \xi_j - \partial_j \xi_i$ to get:

$$\begin{aligned} \tilde{\phi} = \phi - \partial_t A \;\; &; \;\; \tilde{\beta}_\alpha = \beta_\alpha - \partial_t B_\alpha \\ \tilde{\gamma} = \gamma - A - \partial_t C \;\; &; \;\; \tilde{H} = H - 2\nabla^2 C \\ \tilde{\lambda} = \lambda - 2C \;\; &; \;\; \tilde{\epsilon}_\alpha = \epsilon_\alpha - 2B_\alpha \\ & \qquad\qquad \tilde{h}^{TT}_{\alpha\beta} = h^{TT}_{\alpha\beta} \end{aligned}$$

Note that $h^{TT}_{\alpha\beta}$ is gauge invariant.

**Poisson gauge:**

One may remove two scalars and one divergenceless vector by choosing $A$, $C$ and $B_i$. One can choose to remove $\gamma$, $\lambda$ and $\epsilon_i$. This gives the Poisson gauge.

$$h_{tt} = 2\phi \quad ; \quad h_{t\alpha} = \beta_\alpha \ (\partial_\alpha \beta^\alpha = 0)$$
$$h_{\alpha\beta} = \frac{1}{3} H \delta_{\alpha\beta} + h^{TT}_{\alpha\beta}$$

with $\partial_\alpha h^{\alpha\beta\, TT} = 0$, $h^{\alpha\, TT}_{\alpha} = 0$

No residual gauge freedom. Coordinates fixed completely.

## Bardeen's gauge invariant variables:

$$\psi = -\phi + \partial_t \gamma - \frac{1}{2}\partial_t^2 \lambda$$

$$\theta = \frac{1}{3}\left(H - \nabla^2 \lambda\right)$$

$$\Sigma_\alpha = \beta_\alpha - \frac{1}{2}\partial_t \epsilon_\alpha$$

In addition, there is $h_{\alpha\beta}^{TT}$, which is gauge invariant. Total 6.

Bardeen variables reduce to $-\phi$, $H$ and $\beta_\alpha$ and $h_{\alpha\beta}^{TT}$ in the Poisson gauge.

Express the Einstein equations in terms of the Bardeen variables.

**Einstein tensors:**

$$G_{tt} = -\nabla^2\theta \quad ; \quad G_{t\alpha} = -\frac{1}{2}\nabla^2\Sigma_\alpha - \partial_\alpha\partial_t\theta$$

$$G_{\alpha\beta} = -\frac{1}{2}\Box h^{TT}_{\alpha\beta} - \partial_{(\alpha}\partial_t\Sigma_{\beta)} - \frac{1}{2}\partial_\alpha\partial_\beta\left(2\psi + \theta\right) + \delta_{\alpha\beta}\left[\frac{1}{2}\nabla^2\left(2\psi+\theta\right) - \partial_t^2\theta\right]$$

Using the Einstein equations $G_{ij} = 8\pi T_{ij}$ ($G = c = 1$) we get

$$G_{tt} = 8\pi T_{tt} \rightarrow \nabla^2\theta = -8\pi\rho$$

$$G_{t\alpha} - 8\pi T_{t\alpha} = \left(-\partial_\alpha\partial_t\theta - 8\pi\partial_\alpha S\right) + \left(-\frac{1}{2}\nabla^2\Sigma_i - 8\pi S_i\right) = 0$$

Uniqueness of Helmholtz decomposition implies

$$\partial_t\theta + 8\pi S = 0 = -\tfrac{1}{2}\nabla^2\Sigma_i - 8\pi S_i$$

Apply same procedure to $G_{\alpha\beta} - 8\pi T_{\alpha\beta} = 0$ to get

$$-\frac{1}{2}\left(\Box h^{TT}_{\alpha\beta} + 16\pi\sigma_{\alpha\beta}\right) - \left[\partial_{(\alpha}\partial_t\Sigma_{\beta)} + 8\pi\partial_{(\alpha}\sigma_{\beta)}\right] - \partial_\alpha\partial_\beta\left(\psi + \frac{\theta}{2} + 8\pi\sigma\right) + \delta_{\alpha\beta}\left[\frac{1}{2}\nabla^2(2\psi+\theta) - \partial_t^2\theta + \frac{8}{3}\pi\nabla^2\sigma - 8\pi p\right] = 0$$

This leads to the equations:

$$\Box h^{TT}_{\alpha\beta} = -16\pi\sigma_{\alpha\beta}$$

$$\partial_t\Sigma_\alpha + 8\pi\Sigma_\beta = 0$$

$$\psi + \frac{\theta}{2} + 8\pi\sigma = 0$$

$$\nabla^2\left(\psi + \frac{\theta}{2}\right) - \frac{3}{2}\partial_t^2\theta - 12\pi p = 0$$

Using the energy conservation and Euler eqns from $\partial_i T^{ij} = 0$ we get

$$\nabla^2 S = \frac{\partial \rho}{\partial t} \quad ; \quad \nabla^2 \sigma = -\frac{3}{2} p + \frac{3}{2} \partial_t S \quad ; \quad \nabla^2 \sigma_\alpha = 2 \partial_t S_\alpha$$

This finally leads to

$$\begin{aligned} \nabla^2 \theta &= -8\pi\rho \\ \nabla^2 \psi &= 4\pi(\rho + 3p - 3\partial_t S) \\ \nabla^2 \Sigma_\alpha &= -16\pi S_\alpha \\ \Box h^{TT}_{\alpha\beta} &= -16\pi \sigma_{\alpha\beta} \end{aligned}$$

**Six eqns. for six gauge invariant variables. Only one wave eqn for the two independent components of**

$$h^{TT}_{\alpha\beta}$$

## Quadrupole formula: other derivations

**Dimensional arguments:**

Any matter source can be characterised by the following moments.

$$Mass\ monopole : M = \int \rho \, d^3x$$

$$Mass\ Dipole : D = \int \rho \, x^i d^3x$$

$$Mass\ Quadrupole : Q_{ij} = \int \rho \, x_i x_j d^3x$$

$Angular\ momentum\,(moment\ of\ mass\ current) :$

$$L_i = \int \rho \, \epsilon_{ijk} \, x^j v^k d^3x$$

Metric perturbations $h$ are dimensionless. Use dimensional analysis to show the signal which may be constructed.

With $M$. the possible one is $h \sim \frac{GM}{c^2 r}$

With $D$, it is $h \sim \frac{G\dot{D}}{rc^3} \sim \frac{GP}{rc^3}$

With $L_i$ it is $h \sim \frac{G\dot{L}}{rc^4}$

**These are all zero or static due to conservation of mass, momentum and angular momentum.**

**The quadrupole term gives $h \sim \frac{G\ddot{Q}}{rc^4}$. There is no conservation law here.**

Note that in theories beyond GR, radiation sourced by mass monopole, dipole and angular momentum is possible due to exchanges with degrees of freedom other than gravitons.

**Pseudotensor approach:**

*A pseudotensor is an object which transforms as a tensor under linear transformations but not under more general coordinate transformations.*

Problem with the standard derivation of quadrupole formula: (i) weak gravity assumption (ii) use of conservation law in flat spacetime.

Harmonic gauge: $\Box x^i = 0$ (coordinates transform as scalars) can be rewritten using

$$\hat{H}^{ij} = \eta^{ij} - \sqrt{-g} g^{ij}$$

.

$$\Box x^k = \frac{1}{\sqrt{-g}} \partial_i \left( \sqrt{-g}\, g^{ij} \partial_j x^k \right) = \frac{1}{\sqrt{-g}} \partial_i \left( \sqrt{-g} g^{ij} \delta_j^k \right)$$

$$\frac{1}{\sqrt{-g}} \partial_i \left( \sqrt{-g} g^{ik} \right) \sim \partial_i \hat{H}^{ij}$$

Hence: $\partial_i \hat{H}^{ij} = 0$. So, $\hat{H}^{ij}$ becomes the trace-reversed metric perturbation and the above condition is the Lorenz gauge condition. In such a gauge, the full Einstein equations are:

$$\Box_\eta \hat{H}^{ij} = -16\pi\tau^{ij}$$

where

$$\tau^{ij} = (-g)T^{ij} + \frac{1}{16\pi}\Lambda^{ij}$$

and

$$\Lambda^{ij} = 16\pi(-g)t^{ij}_{LL} + \left(\partial_k \hat{H}^{im}\partial_m \hat{H}^{jk} - \partial_k\partial_m \hat{H}^{ij}\hat{H}^{mk}\right)$$

$t^{ij}_{LL}$ is the Landau-Lifshitz pseudotensor.

**Landau-Lifshitz pseudotensor $t^{ij}_{LL}$**

$$16\pi(-g)t^{ij}_{LL} = g_{kl}g^{mn}\partial_m\hat{H}^{ik}\partial_n\hat{H}^{jl}$$
$$+\frac{1}{2}g_{kl}g^{ij}\partial_n\hat{H}^{km}\partial_m\hat{H}^{nl} - 2g_{lm}g^{k(i}\partial_n\hat{H}^{jm)}\partial_k\hat{H}^{nl}$$
$$+\frac{1}{8}\left(2g^{ik}g^{jl} - g^{ij}g^{kl}\right)\left(2g_{mn}g_{pq} - g_{np}g_{mq}\right)\partial_k\hat{H}^{mq}\partial_l\hat{H}^{np}$$

- Stress-energy of the gravitational field.
- Nonzero in one frame but vanishing in another. Consequently explains the locally vanishing aspect of gravity.
- Note that $\partial_i\tau^{ij} = 0$. Follow the same path hereon using $\Box_\eta\hat{H}^{ij} = -16\pi\tau^{ij}$ to derive the quadrupole formula.
- One can check that to order $\frac{1}{c^2}$, $\tau^{tt} = T^{tt}$. But not so for $\tau^{\alpha\beta}$!

**Feynman graph derivation:**

A completely different derivation of the quadrupole formula is found in:

**N. D. Hari Dass and Vikram Soni, Feynman graph derivation of the Einstein quadrupole formula, 1982 J. Phys. A: Math. Gen. 15 473**

- Uses the field theory approach and the field theory linearised gravity Lagrangian.
- Does a perturbation theory analysis (all possible contributing Feynman diagrams) to find the one graviton emission transition matrix.
- Uses the Born approximation and calculates the transition matrix.
- Takes the classical limit to get to the quadrupole formula for energy loss.

## Gravitational wave energy: Isaacson tensor and averaging

Reference:

A. Saffer, N. Yunes and K. Yagi, The gravitational wave stress–energy (pseudo)-tensor in modified gravity, Class. Quantum Grav. 35 (2018) 055011

1. See Section 2.2 for details on Isaacson's method.

2. See Appendix A for Brill-Hartle averaging scheme.